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CONSTRUCTION OF EXACT SOLUTIONS OF THE BOUSSINESQ EQUATION

O. V. Kaptsov

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The known model of nonlinear dispersive waves, which was proposed by Boussinesq in the second half of the nineteenth century, is considered. Solutions of the Boussinesq equation, which are expressed via elementary functions and describe wave packets, their interaction between each other and with solitons, and some other structures are obtained. To construct these solutions, Hirota's bilinear representation and differential relations specified by ordinary differential equations with constant coefficients are used.

One of the classical Boussinesq equations [1, 2], which describe shallow-water waves moving in both directions, has the form

$$\eta_{tt} = gh_0\eta_{xx} + \frac{3}{2}g(\eta^2)_{xx} + \frac{1}{3}gh_0^3\eta_{xxxx},$$

where g is the acceleration of gravity, h_0 is the undisturbed depth, and η is the deviation of the water surface from the undisturbed state. After the transformations $\eta' = 2h_0\eta$, $t' = \sqrt{3g/h_0}\,t$, and $x' = \sqrt{3}x/h_0$ the equation reduces to the form

$$\eta'_{t't'} = \eta'_{x'x'} + 3(\eta'^2)_{x'x'} + \eta'_{x'x'x'x'}. \qquad (1)$$

Equation (1) attracts the attention of researchers owing to the fact that it possesses soliton solutions. A formula of N-soliton solutions was derived by Hirota [3], and rational solutions were given in [4]. In the present study, new solutions of the Boussinesq equation are found, and some of them are interpreted from a hydrodynamic point of view. In particular, solutions that describe the propagation of wave packets and their interaction are obtained.

As Hirota noted, it is convenient to begin the construction of solutions of Eq. (1) by reducing it to the bilinear form. To do this, it is necessary to make the replacement

$$\eta' = 2\frac{d^2}{dx'^2}(\ln u)$$

and to integrate twice the resulting equation. Assuming the functions appearing in the integration to be zero, we come to the bilinear equation

$$u_{tt}u - u_t^2 - uu_{xxxx} + 4u_xu_{xxx} - 3u_{xx}^2 - uu_{xx} + u_x^2 = 0. \qquad (2)$$

For the sake of simplicity, the primes in the variables t and x are omitted in (2) and subsequent equations.

As is known [3], one- and two-soliton solutions of Eq. (1) are produced by the following solutions of Eq. (2):

$$u_1 = 1 + s\exp(kx \pm kt\sqrt{1+k^2}\,),$$

$$u_2 = 1 + \exp(k_1x + m_1t + s_1) + \exp(k_2x + m_2t + s_2)$$
$$+ p_{12}\exp((k_1 + k_2)x + (m_1 + m_2)t + s_1 + s_2),$$

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where s, s_1, s_2, k, k_1, and k_2 are arbitrary constants, $m_i = \pm k_i\sqrt{1+k_i^2}$, and

$$p_{12} = \frac{3(k_1-k_2)^2+(n_1-n_2)^2}{3(k_1+k_2)^2+(n_1-n_2)^2}, \qquad n_i = m_i/k_i. \tag{3}$$

It is noteworthy that the functions u_1 and u_2 satisfy the fourth-order ordinary differential equation

$$d_x(d_x-k_1)(d_x-k_2)(d_x-k_1-k_2)u = 0, \tag{4}$$

where d_x is a derivative with respect to x. It turns out that there are other solutions of this equation producing solutions of the Boussinesq equation.

The system formed by Eqs. (2) and (4) is integrated in two steps. We choose the constants k_1 and k_2 at the first step and find the general solution of Eq. (4) at the second. The variable t enters into this solution parametrically. To determine the dependence of the function u on time, the general solution obtained is substituted into Eq. (2) at the second step. As a result, we obtain an overdetermined system of ordinary differential equations. By solving this system, we find the function $u(t,x)$ producing the solution of the Boussinesq equation. We shall illustrate this scheme by examples.

We consider the simplest case: $k_1 = k_2 = 0$. The solution of Eq. (4) is the polynomial $u = r_3x^3 + r_2x^2 + r_1x + r_0$. Here r_i can be t-dependent. With the polynomial substituted into Eq. (2), it is easy to obtain two known solutions:

$$u = (x\pm t)^3 + (x \pm t) \mp 6t, \qquad u = x^2 - t^2 - 3.$$

If $k_1 = 0$ and $k_2 = k \neq 0$, the general solution of Eq. (4) is $u = (r_3x + r_2)\exp(kx) + r_1x + r_0$. After a substitution of the given function u into (2), we find r_i:

$$r_3 = 1, \qquad r_2 = t, \qquad r_1 = s\exp(mt)(1+\sqrt{1+k^2}+2k^2)/6k,$$

$$r_0 = s\exp(mt)(1+t(1+\sqrt{1+k^2}+2k^2)/6k),$$

where s is an arbitrary constant and $m = k\sqrt{1+k^2}$.

For $k_1 = k_2 = k \neq 0$, it is possible to find the following solution of Eqs. (2) and (4):

$$u = \exp(2kx - 2mt) + \exp(kx - mt)(r_1x + r_2) + r_0,$$

where $r_1 = s$, $r_2 = -st(2k^2+1)/\sqrt{1+k^2}$, $r_0 = -s^2(4k^2+3)/12k^2(1+k^2)$, s is an arbitrary constant, and $m = \sqrt{k^2+k^4}$.

In the case of purely imaginary constants $k_1 = ik$ and $k_2 = -ik$, there are two classes of solutions of Eqs. (2) and (4). The first class is formed by functions that generate singular solutions of Eq. (1), and the second by those producing regular solutions of this equation. To the first class we attribute the following solutions:

$$u = \sin(kx - mt) + ax + bt, \qquad u = \sin(kx) + c_1\cos(mt) + c\sin(mt).$$

Here

$$m = \sqrt{k^2-k^4}, \quad a = \sqrt{\frac{3m^2}{3-4k^2}}, \quad b = \frac{a(2k^2-1)}{\sqrt{1-k^2}}, \quad c_1 = \sqrt{\frac{1-c^2+k^2c^2-4k^2}{1-k^2}}, \quad c \in R.$$

The second class is represented by the solution

$$u = \sin(kx) + \exp(t\sqrt{k^4-k^2}) + \frac{4k^2-1}{4(k^2-1)}\exp(-t\sqrt{k^4-k^2}). \tag{5}$$

A very unusual solution of the Boussinesq equation shown in Fig. 1 for $k = 1.5$ corresponds to function (5). This solution is periodic in x, and its amplitude tends exponentially rapidly to zero for $t \to \pm\infty$. The specifics of this solution is that, following Lighthill [5, p. 299], a wave arises "from nothing" during a short time interval and then damps rapidly.

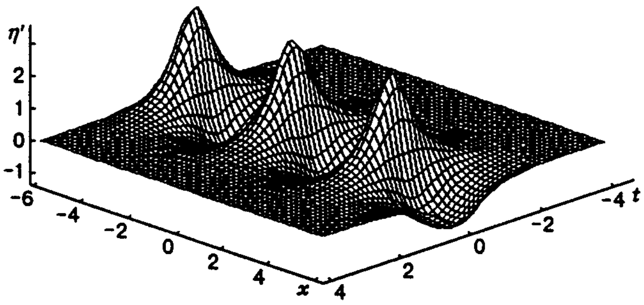

Fig. 1

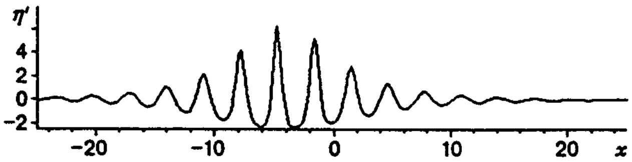

Fig. 2

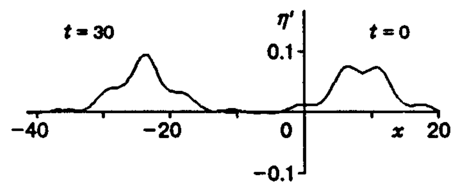

Fig. 3

Of special interest is the case of conjugate complex numbers $k_1 = a + ib$ and $k_2 = a - ib$, the solution of Eqs. (2) and (4) being of the form

$$u = 1 + 2\cos(bx + qt)\exp(ax + rt) + p_{12}\exp(2ax + 2rt). \tag{6}$$

Here r and q are the real and imaginary parts of the number $m = \sqrt{k^2 + k^4}$, and p_{12} is determined by formula (3). By analogy with the Korteweg–de Vries and sine-Gordon modified equation [4, 6], we call the solution of Eq. (1) that corresponds to function (6) a breezer. The behavior of the breezer is determined by the numbers a and b. If we take $a = 0.2$ and $b = 2$, the wave packet moving with group velocity $v_{gr} = -r/a = 17.1$ will correspond to solution (6). The breezer for $t = 0$ is shown in Fig. 2. No extension of the wave packet occurs with time. It is possible to decrease the amplitude of the wave packet (the maximum amplitude of the waves forming the wave packet) by decreasing the parameter b and leaving a unchanged. For example, the amplitude is somewhat higher than 0.4 for $b = 1$, and it does not exceed 0.2 for $b = 0.95$. A further decrease in the parameter b leads to restructuring of the packet in a solitonlike structure over which the waves ("ripples") run. We shall call this structure a solitonlike breezer. Figure 3 shows the breezer at the moments of time $t = 0$ and $t = 30$ ($b = 0.933$). If the parameter b is further decreased, the solution of Eq. (1), which corresponds to the function (6), becomes discontinuous. It is necessary to note that breezer-type smooth solutions were not found for the Korteweg–de Vries equation.

It is easy to write an ordinary differential equation satisfied by a function producing a three-soliton solution of the equation

$$d_x(d_x - k_1)(d_x - k_2)(d_x - k_3)(d_x - k_1 - k_2)(d_x - k_1 - k_3)(d_x - k_2 - k_3)(d_x - k_1 - k_2 - k_3)u = 0. \tag{7}$$

By choosing the constants k_i in various ways, by finding the general solution of Eq. (7), by substituting it into (2), and by integrating second-order ordinary equations, one can obtain various solutions of the Boussinesq equation.

We shall give a solution that describes the elastic soliton-breezer interaction. This solution arises in the case where k_1 and k_2 are conjugate complex numbers, and k_3 is a real number. The corresponding function u has the form

$$u = 1 + 2\cos(bx + qt)\exp(ax + rt) + p_{12}\exp(2ax + 2rt) + \exp(k_3 x + m_3 t)$$

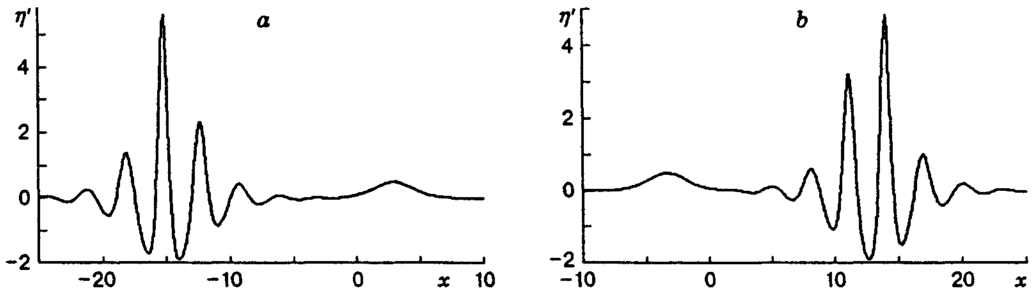

Fig. 4

$$+ 2(p_{13r}\cos(bx+qt) - p_{13i}\sin(bx+qt))\exp((a+k_3)x+(r+m_3)t)$$
$$+ |p_{13}|^2 p_{12}\exp((2a+k_3)x+(2r+m_3)t).$$

Here p_{12} is given by formula (3), and p_{13} is given by the same formula with replacement of the subscript 2 by 3; a and b, r and q, and p_{13r} and p_{13i} denote the real and imaginary parts of the complex numbers k_1, $m = k_1\sqrt{1+k_1^2}$, and p_{13}, respectively. Figure 4a and b show the wave packet and the soliton before and after the interaction. It is possible to obtain a solution that describes the soliton-solitonlike breezer interaction by a choice of the free parameters: $a = 0.5$, $b = 1.23$, and $k_3 = 1$.

Various exact solutions of the Boussinesq equation can be found using Eqs. (7) and its higher-order analogs. Omitting the concrete form of the function, we note that a solution that describes the interaction of two breezers is derived from the expression for a 4-soliton solution [3] if the pairs of numbers k_1 and k_2 and k_3 and k_4 are taken as conjugate complex ones. The problem of the stability of the solutions obtained still remains open. Of interest is performing experiments with a view for studying breezer solutions.

In concluding, we mention that the cited scheme of constructing exact solutions is valid for other equations that admit Hirota's bilinear representation, the structure of the differential relations (4) and (7) remaining the same.

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REFERENCES

1. P. Bhatnagar, *Nonlinear Waves in One-Dimensional Dispersive Systems*, Clarendon Press, Oxford (1979).
2. G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley and Sons, New York (1974).
3. R. Hirota, "Exact N-soliton solution of the wave equation of long waves in shallow water and in nonlinear lattices," *J. Math. Phys.*, **14**, 810–814 (1973).
4. M. Ablowitz and H. Segur, *Solitons and Inverse Scattering Transform*, SIAM, Philadelphia (1981).
5. J. Lighthill, *Waves in Fluids*, Cambridge Univ. Press, Cambridge (1978).
6. G. L. Lamb, Jr., *Elements of Soliton Theory*, John Wiley and Sons, New York (1980).